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Stability of Two-Dimensional
 Recursive Filters

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STABILITY OF TWO-DIMENSIONAL RECURSIVE FILTERS

T. S. HUANG

Consultant, Group 64

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ABSTRACT

We discuss some aspects of the stability problem in two-dimensional recursive filtering. In particular, we derive a simplified version of a stability theorem due to Shanks and show that it is equivalent to some results of Ansell. We also give several examples of stability tests and pose a few unsolved problems.

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CONTENTS

Abstract	iii
I. Introduction	1
II. Shanks Stability Theorem	4
a) Statement of the Theorem	5
b) Proof	5
c) Test Procedure	6
d) An Implication	6
III. A Simplified Version of Shanks Theorem	9
a) The Result	9
b) Proof	9
c) Test Procedure	10
IV. Ansell Stability Theorem	11
V. Examples	13
a) First-order Filters	13
b) A Special Class of Second-order Filters	14
c) A Numerical Example	17
VI. Concluding Remarks	18
References	20

STABILITY OF TWO-DIMENSIONAL RECURSIVE FILTERS

I. Introduction

In digital image processing, we often want to do linear filtering. The use of recursive instead of non-recursive filters has the potential of saving computation time. A two-dimensional digital recursive filter is characterized by the two-dimensional z-transform

$$H(z_1, z_2) = \frac{\sum_{m=0}^p \sum_{n=0}^q a_{mn} z_1^m z_2^n}{\sum_{m=0}^p \sum_{n=0}^q b_{mn} z_1^m z_2^n} \quad (1)$$

where a_{mn} and b_{mn} are constants, and without loss of generality we can set $b_{00} = 1$. The degrees of the numerator and the denominator polynomials do not have to be equal, since some of the coefficients a_{mn} and b_{mn} can be zero. The variables z_1 and z_2 are defined as

$$\begin{cases} z_1 = e^{-s_1 A} \\ z_2 = e^{-s_2 B} \end{cases} \quad (2)$$

where s_1 and s_2 are respectively the horizontal and vertical complex spatial frequency variables and A and B are constants (sampling periods in the horizontal and vertical directions, respectively).

Let $f(m, n)$ and $g(m, n)$ be the input and output, respectively of the filter. Then the spatial-domain difference equation corresponding to Eq. (1) is

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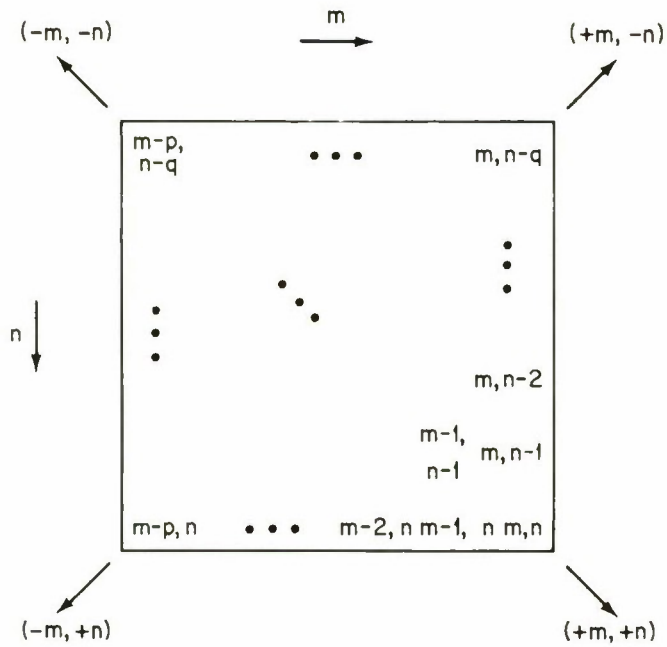
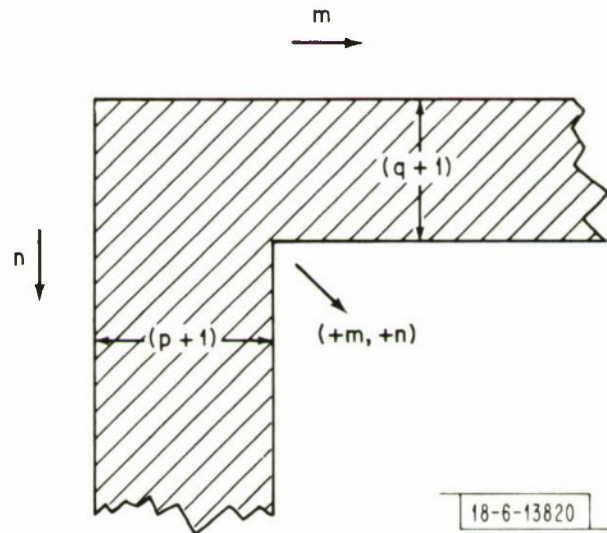


Fig. 1. Four directions of recursion.

Fig. 2. Initial conditions for a casual filter.



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$$\sum_{m=0}^p \sum_{n=0}^q b_{mn} g(M-m, N-n) = \sum_{m=0}^p \sum_{n=0}^q a_{mn} f(M-m, N-n) \quad (3)$$

We can express $g(M, N)$ in terms of the rest, and thus obtain a recursive filter which recurses in the $(+m, +n)$ direction. We can also express $g(M-p, N-q)$ in terms of the rest, and obtain a recursive filter which recurses in the $(-m, -n)$ direction. Similarly, we can get from Eq. (3) recursive filters recursing in the $(+m, -n)$ and the $(-m, +n)$ directions. See Fig. 1. We shall call the recursive filter that recurses in the $(+m, +n)$ direction causal. Thus, the z -transform of Eq. (1) can be associated with four different recursive filters. However, unless otherwise specified, we shall associate it with the causal one.

To start recursing, we need initial conditions. For the causal filter, we need to know the values of the output g in the shaded region in Fig. 2.

There are two major problems in the design of a recursive filter: approximation and stability. The approximation problem consists of determining the coefficients a_{mn} and b_{mn} in Eq. (1) so that $H(e^{-juA}, e^{-jvB})$ approximates a given frequency response $H_1(ju, jv)$, where u and v are respectively the horizontal and vertical spatial frequencies. In order for the resulting filter $H(z_1, z_2)$ to be useful, we have to require that it be stable, i.e., if we expand $H(z_1, z_2)$ into a power series

$$H(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h_{mn} z_1^m z_2^n \quad (4)$$

the coefficients, h_{mn} , which is the impulse response of the filter, should satisfy

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |h_{mn}| < \infty \quad (5)$$

If the filter is unstable, any noise (including the roundoff errors in computation) will propagate through the output and possibly be amplified. Furthermore, in many important applications, such as inverse filtering for image restoration, the initial conditions are usually not known. We, therefore, desire that the result of the recursive filtering be roughly independent of the initial conditions. This will be true if the impulse response $|h_{mn}| \approx 0$ for $m, n > A$, where A is a constant which is much smaller than the linear dimensions of the input and output images. Therefore, in some cases, stability itself is not sufficient; we require in addition that the impulse response die out fast enough.

In this paper, we shall discuss some aspects of the stability problem of two-dimensional recursive filters. In particular, we shall derive a simplified version of Shanks stability theorem^{1,2}, and show that it is equivalent to Ansell stability theorem^{3,4}.

II. Shanks Stability Theorem

Both the theorems of Shanks and Ansell tell us stability conditions in the frequency domain. In this section, we shall state and prove Shanks stability theorem and discuss its implications.

a) Statement of the Theorem

Theorem 1 (Shanks)

A causal recursive filter with the z-transform $H(z_1, z_2) = A(z_1, z_2)/B(z_1, z_2)$, where A and B are polynomials in z_1 and z_2 , is stable, if and only if there are no values of z_1 and z_2 such that $B(z_1, z_2) = 0$ and $|z_1| \leq 1$ and $|z_2| \leq 1$.

b) Proof

Let

$$H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h_{mn} z_1^m z_2^n \quad (6)$$

What we want to show is that $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |h_{mn}| < \infty$ if and only if $H(z_1, z_2)$ is analytic in the region $D = \{(z_1, z_2); |z_1| \leq 1 \cap |z_2| \leq 1\}$.

The "if" part: If $H(z_1, z_2)$ is analytic in D, we can find $\epsilon > 0$ such that $H(z_1, z_2)$ is analytic in $D_1 = \{(z_1, z_2); |z_1| < 1 + \epsilon \cap |z_2| < 1 + \epsilon\}$, which implies that

$$\sum_m \sum_n h_{mn} z_1^m z_2^n \text{ is absolutely convergent in } D_1.$$

Therefore $\sum_m \sum_n |h_{mn}| < \infty$.

The "only if" part: If $\sum_m \sum_n |h_{mn}| < \infty$, then by the M-test,

$\sum_m \sum_n h_{mn} z_1^m z_2^n$ is absolutely convergent in D, which implies in turn that $H(z_1, z_2)$ is analytic in D. Q.E.D.

c) Test Procedure

To test stability using Shanks theorem is hard work. One way, e.g., is to map the unit disk $d_1 \equiv (z_1, |z_1| \leq 1)$ in the z_1 -plane into the z_2 -plane by the implicit mapping relation $B(z_1, z_2) = 0$. The filter is stable if and only if the image of d_1 in the z_2 -plane does not overlap the unit disk $d_2 \equiv (z_2, |z_2| \leq 1)$. This is hard work, because for each particular value of $z_1 = z_1^0 \in d_1$, we have to solve the equation $B(z_1^0, z_2) = 0$ for z_2 . And we have to do that for all (in practice, a large number of points) in d_1 .

d) An Implication

We emphasize that Shanks theorem was stated for causal filters, that is, filters which recurse in the $(+m, +n)$ direction. Let us consider the filter associated with $H(z_1, z_2) = A(z_1, z_2)/B(z_1, z_2)$ which recurses in the $(-m, -n)$ direction. We assume that the orders of A in both z_1 and z_2 are smaller than the corresponding orders of B . Then, the stability condition becomes: The image of $(z_1, |z_1| \geq 1)$ in the z_2 -plane, according to $B(z_1, z_2) = 0$, does not overlap $(z_2, |z_2| \geq 1)$.

This can be shown by expanding $H(z_1, z_2)$ in terms of z_1^{-1} and z_2^{-1} :

$$H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h_{mn}^1 z_1^{-m} z_2^{-n} \quad (7)$$

Making the change of variables

$$\begin{cases} w_1 = z_1^{-1} \\ w_2 = z_2^{-1} \end{cases} \quad (8)$$

We get

$$H(w_1^{-1}, w_2^{-1}) = \frac{A(w_1^{-1}, w_2^{-1})}{B(w_1^{-1}, w_2^{-1})} = \sum_m \sum_n h_{mn}^1 w_1^m w_2^n \quad (9)$$

Now

$$\frac{A(w_1^{-1}, w_2^{-1})}{B(w_1^{-1}, w_2^{-1})} = w_1^a w_2^b \frac{A_1(w_1, w_2)}{B_1(w_1, w_2)} \quad (10)$$

Where a and b are non-negative integers, and $B_1(w_1, w_2) = 0$ if and only if $B(w_1^{-1}, w_2^{-1}) = 0$. Applying Shanks theorem to

$$\frac{w_1^a w_2^b A_1(w_1, w_2)}{B_1(w_1, w_2)} = \sum_m \sum_n h_{mn}^1 w_1^m w_2^n \quad (11)$$

we know that $\sum_m \sum_n |h_{mn}^1| < \infty$ if and only if the image of $(w_1, |w_1| \leq 1)$ in the w_2 -plane, according to $B_1(w_1, w_2) = 0$, does not overlap $(w_2; |w_2| \leq 1)$. But $w_1 = z_1^{-1}$, and $w_2 = z_2^{-1}$. So our assertion has been proved.

Similarly, we can get the stability conditions for the filters associated with $H(z_1, z_2)$ that recurse in the $(+m, -n)$ and the $(-m, +n)$ directions. We can easily convince ourselves that at most one of the four stability conditions, corresponding to the four recursing directions, can be satisfied. Therefore, we have the following Theorem 2

Among the four recursive filters we can associate with the z -transform of Eq. (1), or equivalently the difference equation of Eq. (3), at most one can be stable.

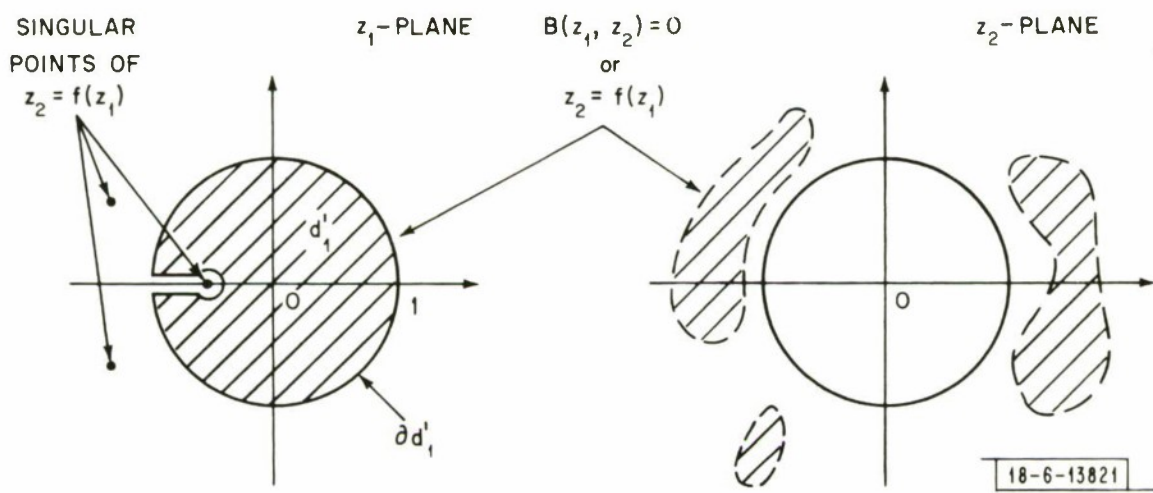


Fig. 3. The mapping $z_2 = f(z_1)$.

III. A Simplified Version of Shanks Theorem

Shanks' result as stated in Theorem 1 is very tedious to apply. We now show that his stability conditions can be simplified considerably.

a) The Result

Theorem 3

A causal filter with a z -transform $H(z_1, z_2) = A(z_1, z_2)/B(z_1, z_2)$, where A and B are polynomials, is stable, if and only if: (1) the map of $\partial d_1 \equiv (z_1; |z_1| = 1)$ in the z_2 -plane, according to $B(z_1, z_2) = 0$, lies outside $d_2 \equiv (z_2; |z_2| \leq 1)$; and (2) no point in $d_1 = (z_1; |z_1| \leq 1)$ maps into the point $z_2 = 0$ by the relation $B(z_1, z_2) = 0$.

b) Proof

We want to establish that the stability conditions of Theorems 1 and 3 are equivalent. That the stability conditions of Theorem 1 imply those of Theorem 3 is obvious. So we proceed to show the implication in the other direction.

The two-variable polynomial $B(z_1, z_2) = 0$ defines an algebraic function⁵ $z_2 = f(z_1)$. We first modify the unit-circle contour in the z_1 -plane to exclude any singular points of f inside the contour, resulting in a modified contour $\partial d'_1$ as shown in Fig. 3. We use d'_1 to denote the closed region enclosed by $\partial d'_1$. A point $z_1 = z_1^0$ is called a singular point of $z_2 = f(z_1)$, if $B(z_1^0, z_2) = 0$, considered as an equation in z_2 , has multiple (finite or infinite) roots.

According to the theory of algebraic functions, in d'_1 , the function $z_2 = f(z_1)$ has a number of branches, each of which is holomorphic. Therefore, from the maximum-modulus theorem, the maximum of $|f(z_1)|$ over d'_1 occurs on $\partial d'_1$, and the minimum of $|f(z_1)|$ over d'_1 can occur in the interior only if the minimum is zero. However, condition (2) of Theorem 3 says $f(z_1)$ is never zero in d'_1 . Therefore, the minimum of $f(z_1)$ occurs on $\partial d'_1$, i.e.,

$$|f(d'_1)| \geq \min. |f(\partial d'_1)| \quad (12)$$

which implies that: if $|f(\partial d'_1)| > 1$, then $|f(d'_1)| > 1$; i.e., to ensure that $f(d'_1)$ lies outside the unit disk $d_2 = (z_2, |z_2| \leq 1)$, it is sufficient to ensure that $f(\partial d'_1)$ lies outside d_2 .

We are almost there, but not quite. What we really want to show is that: if $|f(\partial d_1)| > 1$, then $|f(d_1)| > 1$, where $d_1 \equiv (z_1, |z_1| \leq 1)$ and $\partial d_1 \equiv (z_1; |z_1| = 1)$. Since the detour in $\partial d'_1$ can be any path leading from ∂d_1 to the singular point, what is left to show is simply that $|f(s)| > 1$ where s is the singular point. But since each branch of $z_2 = f(z_1)$ is continuous at $z_1 = s$, and since $|f(s + \epsilon e^{j\theta})| > 1$ for arbitrary small ϵ and any θ , we have $|f(s)| > 1$. Q.E.D.

c) Test Procedure

To test stability using Theorem 3, we have to map $\partial d_1 = (z_1; |z_1| = 1)$ into the z_2 -plane according to $B(z_1, z_2) = 0$, and see whether the image lies outside $d_2 \equiv (z_2; |z_2| \leq 1)$. Also, we have to solve $B(z_1, 0) = 0$ to see whether there is any root with magnitude smaller than 1.

IV. Ansell Stability Theorem

To test the stability of a two-dimensional recursive filter using Theorem 3 is much simpler than using Shanks' original Theorem 1. However, the procedure is still infinite in the sense that in principle we have to map every point on the continuum ∂d , into the z_2 -plane. In this section, we show that Theorem 3 can be reduced to a result due to Ansell. We shall see that Ansell's result enables us to test stability in a finite number of steps - which, unfortunately, can still be very tedious.

By making the change of variables:

$$p_1 = \frac{1 - z_1}{1 + z_1} \quad (13a)$$

and

$$p_2 = \frac{1 - z_2}{1 + z_2} \quad (13b)$$

and let

$$H(z_1, z_2) = \frac{E(p_1, p_2)}{F(p_1, p_2)} \quad (14)$$

where E and F are polynomials in p_1 and p_2 . We can restate Theorem 3 as:

Theorem 4 (Ansell)

The causal recursive filter $H(z_1, z_2)$ is stable, if and only if: (1) for all real finite w , the complex polynomial in p_2 , $F(jw, p_2)$, has no zeros in $\text{Re } p_2 \geq 0$; and (2) the real polynomial in p_1 , $F(p_1, 1)$, has no zeros in $\text{Re } p_1 \geq 0$.

This theorem is essentially the same as Lemma 6 of Ansell³. The nice thing about this theorem is that condition (1) can be tested using standard techniques of circuit theory⁶.

Theorem 5 (Ansell)

Condition (1) of Theorem 4 is equivalent to the following: Let us express $F(jw, j\Omega)$, where w and Ω are real, as

$$F(jw, j\Omega) = b_0(w) \Omega^n + b_1(w) \Omega^{n-1} + \dots + b_n(w) + j \left[a_0(w) \Omega^n + a_1(w) \Omega^{n-1} + \dots + a_n(w) \right] \quad (15)$$

where $a_i(w)$ and $b_i(w)$ are real polynomials in w and either $a_0(w)$ or $b_0(w)$ is not identically zero. Let $C_{r,s}(w)$ be defined as

$$C_{r,s} = a_r b_s - a_s b_r \quad (16)$$

for $0 \leq r, s \leq n$ (using zero for a 's and b 's not present in $F(jw, j\Omega)$). And let $D(w)$ denote the $n \times n$ symmetrical polynomial matrix whose typical element $D_{ij}(w)$ ($1 \leq i, j \leq n$) is the sum of all those $C_{r,s}(w)$ ($0 \leq r, s \leq n$) for which both

$$s + r = i + j - 1 \quad (17a)$$

and

$$s - r > |i - j| \quad (17b)$$

are satisfied. Then, the n successive principal minors of $D(w)$ must be positive for all real w .

This is part of Theorem II of Ansell^{3,7}. We note that Sturm's method⁶ can be used to test whether each minor of $D(w)$ is positive for all real w .

V. Examples

We now give some examples illustrating the use of Theorem 3 and Theorems 4 and 5. Ideally, we would like to determine what relations the coefficients a_{ij} and b_{ij} in Eq. (1) must satisfy to ensure the stability of a causal recursive filter $H(z_1, z_2)$. Unfortunately, this does not seem possible except for very simple filters.

a) First-order Filters

Let us consider the first-order filter

$$H(z_1, z_2) = \frac{1}{1 + az_1 + bz_2} \quad (18)$$

where a and b are constants. In this simple case, we can easily determine the stability conditions in terms of a and b using Theorem 1.

Let

$$B(z_1, z_2) = 1 + az_1 + bz_2 = 0$$

we have

$$z_2 = -\frac{1}{b} - \frac{a}{b} z_1 \quad (19)$$

Therefore, $B = 0$ maps the unit disk in the z_1 -plane into a disk in the z_2 -plane with center at $z_2 = -\frac{1}{b}$ and radius $|\frac{a}{b}|$. This image will not overlap the unit disk in the z_2 -plane, if and only if

$$\left| \frac{1}{b} \right| - \left| \frac{a}{b} \right| > 1 \quad (20)$$

or equivalently,

$$|a| + |b| < 1 \quad (21)$$

We now try to establish the same result by using Theorems 4 and 5. Making the change of variables according to Eqs. (13), we get

$$H(z_1, z_2) = \frac{E(p_1, p_2)}{F(p_1, p_2)} = \frac{1+p_1+p_2+p_1 p_2}{(1+a+b)+(1-a+b) p_1 + (1+a-b) p_2 + (1-a-b) p_1 p_2} \quad (22)$$

Condition (2) of Theorem 4 dictates that $F(p_1, 1) = 2(1+a) + 2(1-a) p_1 = 0$ has not root in $\text{Re } p_1 \geq 0$. Therefore

$$|a| < 1 \quad (23)$$

Now we tackle condition (1) of Theorem 4 with the help of Theorem 5. We have

$$F(jw, j\Omega) = (1+a+b) - (1-a-b) w\Omega + j \left[(1-a+b) w + (1+a-b) \Omega \right]$$

and we want

$$C_{0,1} = (1+a-b)(1+a+b) + (1-a+b)(1-a-b) w^2 > 0, \quad \left\{ \begin{array}{l} \text{for all real } w \end{array} \right. \quad (24)$$

Let $A_1 = 1+a-b$, $A_2 = 1+a+b$, $A_3 = 1-a+b$, and $A_4 = 1-a-b$. Then the inequality (24) is valid if either of the following four conditions is satisfied: (i) $A_1 > 0$, $A_2 > 0$, $A_3 > 0$, $A_4 > 0$. (ii) $A_1 > 0$, $A_2 > 0$, $A_3 < 0$, $A_4 < 0$. (iii) $A_1 < 0$, $A_2 < 0$, $A_3 > 0$, $A_4 > 0$. (iv) $A_1 < 0$, $A_2 < 0$, $A_3 < 0$, $A_4 < 0$. It is easy to show that (i) is equivalent to the inequality (21), while (ii) - (iv) are incompatible with the inequality (23).

b) A Special Class of Second-order Filters

We next consider the filter

$$H(z_1, z_2) = \frac{1}{1+az_1+bz_2+cz_1 z_2} \quad (25)$$

which we shall call a bilinear filter. We first establish stability conditions in terms of the coefficients a , b , and c , using Theorem 3.

Let

$$B(z_1, z_2) = 1 + az_1 + bz_2 + cz_1 z_2 = 0$$

We have

$$z_2 = \frac{1 + az_1}{b + cz_1} \quad (26)$$

which is a bilinear transformation mapping circles into circles. The image of the unit circle $\partial d_1 \equiv (z_1; |z_1| = 1)$ in the z_2 -plane is then a circle. From Eq. (26), the center of this image circle is on the real-axis, and it intersects the real-axis at

$$z_2 = -\frac{1-a}{b-c} \quad \text{and} \quad z_2 = -\frac{1+a}{b+c}$$

It is easy to see that condition (1) of Theorem 3 is satisfied if

$$\left| \frac{1-a}{b-c} \right| > 1 \quad (27a)$$

and

$$\left| \frac{1+a}{b+c} \right| > 1 \quad (27b)$$

Finally, condition (2) of Theorem 3 is equivalent to

$$|a| < 1 \quad (27c)$$

The conditions (27) agree with those of Shanks¹.

We now try to establish the stability conditions using Theorems 4 and 5.

Making the change of variables according to Eqs. (13), we get

$$H(z_1, z_2) = \frac{E(p_1, p_2)}{F(p_1, p_2)} = \frac{1 + p_1 + p_2 + p_1 p_2}{B_1 + B_2 p_1 + B_3 p_2 + B_4 p_1 p_2} \quad (28)$$

where

$$\left\{ \begin{array}{l} B_1 = 1+a+b+c \\ B_2 = 1-a+b-c \\ B_3 = 1+a-b-c \\ B_4 = 1-a-b+c \end{array} \right. \quad (29)$$

For condition (2) of Theorem 4, we form

$$F(p_1, 1) = 2(1+a) + 2(1-a)p_1 = 0$$

whence $\text{Re } p_1 > 0$ means

$$|a| < 1 \quad (30)$$

Now we tackle condition (1) of Theorem 4. We have

$$F(jw, j\Omega) = B_1 - B_4 w\Omega + j[B_2 w + B_3 \Omega]$$

According to Theorem 5, we want

$$\left\{ \begin{array}{l} C_{0,1} = B_3 B_1 + B_2 B_4 w^2 > 0 \\ \text{for all real } w \end{array} \right. \quad (31)$$

which is valid if either of the following four conditions is satisfied: (i) $B_1 > 0$,

$B_3 > 0$, $B_2 > 0$, $B_4 > 0$. (ii) $B_1 > 0$, $B_3 > 0$, $B_2 < 0$, $B_4 < 0$. (iii) $B_1 < 0$, $B_3 < 0$,

$B_2 > 0$, $B_4 > 0$. (iv) $B_1 < 0$, $B_3 < 0$, $B_2 < 0$, $B_4 < 0$.

It is easy to show that (i) is equivalent to

$$|a+b| - 1 < c < 1 - |a-b| \quad (32)$$

and (ii) - (iv) are ruled out by the inequality (30). It can readily be verified that

(32) is equivalent to (27).

c) A Numerical Example

Unfortunately, filters described by Eqs. (18) and (25) are the only types for which we were able to derive stability conditions in terms of the coefficients in the z -transforms. For more complicated filters, the application of Theorem 3 becomes unwieldy. However, for any filter with numerical coefficients, we can always apply Theorems 4 and 5 and determine whether it is stable or not in a finite number of steps. We illustrate by an example.

Consider the filter

$$H(z_1, z_2) = \frac{1}{1 + \frac{1}{2} z_1 + \frac{1}{2} z_2 + \frac{1}{4} z_1 z_2 + \frac{1}{4} z_1^2 + \frac{1}{4} z_2^2} \quad (33)$$

Making the change of variables according to Eqs. (13), and proceeding as before, we find that

$$F(p_1, 1) = 3p_1^2 + 6p_1 + 7$$

whose zeros have negative real parts. Therefore, condition (2) of Theorem 4 is satisfied.

For condition (1) of Theorem 4, we form

$$C_{0,1}(w) = \frac{3}{4} w^4 - \frac{3}{2} w^2 - \frac{15}{4}$$

which can certainly be negative for some real values of w . Therefore, condition (1) is violated. The filter of Eq. (33) is unstable.

VI. Concluding Remarks

We have discussed in this paper how to test the stability of two-dimensional recursive filters in the frequency domain. In conclusion, we would like to point out that what we have discussed is only part of the stability problem. Furthermore, it is not the most important part. The more important questions we want to ask are: (1) How do we design filters that are guaranteed to be stable? and (2) If a given filter is unstable, how do we stabilize it without changing its frequency response?

One approach to attacking question (1) is to study what types of frequency responses we can get from classes of simple filters whose stability we know how to control.

With respect to question (2), we might mention a conjecture of Shanks². Let $H(z_1, z_2) = 1/B(z_1, z_2)$ be an unstable filter. We first determine a least mean-square inverse of $B(z_1, z_2)$, which we call

$$C(z_1, z_2) = \sum_{m=0}^a \sum_{n=0}^b c_{mn} z_1^m z_2^n \quad (34)$$

where a and b are positive integers.

Let

$$B(z_1, z_2) C(z_1, z_2) = 1 + d_{10} z_1 + d_{01} z_2 + d_{11} z_1 z_2 \dots$$

Then among all polynomials of order (a, b) , C is the least mean-square of B if the coefficients of C are such that $(d_{10}^2 + d_{01}^2 + d_{11}^2 + \dots)$ is minimized. We next determine a least mean-square inverse of $C(z_1, z_2)$ which we call $\hat{B}(z_1, z_2)$. Then Shanks conjecture states that the filter $\hat{H}(z_1, z_2) = 1/\hat{B}(z_1, z_2)$ is stable and that the magnitude of the frequency response of \hat{H} is approximately equal to that of H .

The approximation becomes better when we use larger values for a and b in Eq. (34). It was proven by Robinson^{8,9} that a one-dimensional version of Shanks' procedure does yield stable (one-dimensional) filters. However, whether the procedure yields stable filters in two dimensions is still an open question. Also, no analysis is available on how close the magnitude of the frequency response of \hat{H} is to that of H .

We have yet another open question with respect to question (2). In the one-dimensional case, if we are doing non-real-time filtering, then we can always decompose an unstable filter with no poles on the unit circle into two stable ones recursing in opposite directions. We associate the poles of the original filter outside the unit circle with a filter recursing in the positive direction, and the poles inside the unit circle with a filter recursing in the negative direction. Is there an analogous procedure in two dimension? It beats me.

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13. ABSTRACT We discuss some aspects of the stability problem in two-dimensional recursive filtering. In particular, we derive a simplified version of a stability theorem due to Shanks and show that it is equivalent to some results of Ansell. We also give several examples of stability tests and pose a few unsolved problems.		
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